

Dissipative Dynamics and Phase Transitions in Fermionic Systems

Birger Horstmann^{1,2}, J. Ignacio Cirac¹, and Géza Giedke^{1,3}

(1) *Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Straße 1, 85748 Garching, Germany*

(2) *Deutsches Zentrum für Luft- und Raumfahrt, Institut für Technische Thermodynamik, Pfaffenwaldring 38-40, 70569 Stuttgart, Germany and*

(3) *M5, Zentrum Mathematik, Technische Universität München, L.-Boltzmannstr. 3, 85748 Garching, Germany*

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We study abrupt changes in the dynamics and/or steady state of fermionic dissipative systems produced by small changes of the system parameters. Specifically, we consider open fermionic systems whose dynamics is described by master equations that are quadratic (and, under certain conditions, quartic) in creation and annihilation operators. We analyze both phase transitions in steady state, as well as “dynamical transitions”. The latter are characterized by abrupt changes in the rate at which the system asymptotically approaches the steady state. We illustrate our general findings with relevant examples of fermionic (and, equivalently, spin) systems, and show that they can be realized in ion chains.

I. INTRODUCTION

Motivated by the impressive experimental control over many-body quantum states and dynamics [1], open many-body quantum systems have received increasing experimental and theoretical attention in recent years. On the one hand, the decoherence introduced by coupling to an environment is a major challenge to quantum information processing [2], on the other hand, it can play a constructive role for quantum computing [3, 4], state preparation [5–8], entanglement generation [9, 10], quantum memories [11] or quantum simulation [12–16].

These exciting possibilities drive the interest in understanding the steady state phase diagram of open systems in detail [17]. Of particular interest are points of transitions between different phases of the system. For closed systems at zero temperature, the phase diagram and quantum phase transition can be understood by studying the low-lying energy eigenstates of the system’s Hamiltonian [18]. In particular, the non-analyticity of certain expectation values as a function of an external parameter, that characterizes the quantum phase transition, can only occur if the gap of the Hamiltonian closes, i.e., the energy difference between ground state and first excited state vanishes. Quantum phase transitions are thus determined by the low energy spectrum of the Hamiltonian governing the dynamics of wave functions

$$\partial_t |\Phi\rangle = -\frac{i}{\hbar} \mathbf{H} |\Phi\rangle. \quad (1)$$

In this paper we study abrupt changes in the physical properties of a many-body quantum system whose dynamics is described by a master equation

$$\partial_t \rho = \mathcal{S} \rho. \quad (2)$$

This equation describes the dynamics of an open system coupled to a Markovian reservoir, where ρ is the system’s density operator. The superoperator \mathcal{S} contains two parts: one is related to the system Hamiltonian (eventually renormalized due to the interaction with the envi-

ronment) and the other with the dissipation induced by the environment. Under the appropriate conditions, the system evolves to a steady state ρ_{ss} , which corresponds to a (right) eigenstate of \mathcal{S} with eigenvalue 0. Note that this eigenvalue may be degenerate, or there may be other eigenvalues with zero real part, in which case the steady state is not unique. In case this does not happen, the steady state is unique. Then, the other eigenvalues λ of \mathcal{S} have a negative real part, and the smallest absolute value of them, Δ , determines the *asymptotic decay rate* (ADR), that is, the rate at which the steady state is reached. A phase transition in the steady state, where its properties abruptly change when one slightly changes a parameter in the master equation will be accompanied by the vanishing of Δ . This is typically referred to as a “dissipative quantum phase transition” (see, for example, [3, 5, 17, 19, 20]). There is a natural analogy between dissipative and (closed-system) quantum phase transitions: A unique ground state of the Hamiltonian is analogous to a unique steady state. The appearance of a phase transition is signaled by the vanishing of the gap or Δ , respectively.

Apart from its role in reflecting the appearance of a phase transition, the quantity Δ can play an additional role. It also represents a physical property of the system, namely the rate at which the steady state is approached asymptotically or its response to perturbations in the steady state. This quantity may change abruptly itself. In that case, we can talk about a *dynamical transition*, since a small change in the system parameters may lead to an abrupt change of the dynamics of the system. Actually, such a transition may in principle occur even if Δ remains finite, and thus it is a different property than the transitions generally studied in this context.

In this paper we investigate both kinds of transitions for simple fermionic systems. We concentrate on master equations in which the Hamiltonian part is at most quadratic in fermionic creation and annihilation operators. Additionally, we consider two kind of dissipative parts in terms of their dependence on such operators: (i) general quadratic and (ii) quartic, but with some condi-

tions (in particular, that they correspond to Hermitian Lindblad operators). In the first case, the dynamics can be exactly solved [21–24] which has been exploited in several recent works to study the interplay of dissipation and critical Hamiltonians in 1d fermionic systems [19, 23, 25]. In the second case, even though the full dynamics cannot be obtained, we will show that it is nevertheless possible to exactly determine the dynamics of certain expectation values, from which dynamical and steady state properties can be obtained. In this last case we will present analytical examples where dynamical transitions occur [26]. This situation has also been studied in [27–29] with particular regard to transport through a dephasing spin chain, where exact solutions of the associated master equation could be obtained.

Even though the formalism we develop is relatively general, we will illustrate it with explicit examples. In particular, we consider Hamiltonians which are intimately connected to physical situations that can be obtained in the lab, namely anisotropic XY spin chains in transverse magnetic fields, and that a Jordan-Wigner transformation maps it into a fermionic Hamiltonian. This family of Hamiltonians displays the prototype of a continuous phase transition [18]. The dissipative terms we consider can also be understood as particular physical processes occurring in the spin chain through its interaction with an environment. Note that our framework also applies to the systems studied in [21, 22, 27, 28], and for the quadratic dissipative terms is related to [19, 23], where generic dissipative phase transitions are analyzed.

This paper is structured as follows. In Sec. II we introduce the Lindblad master equation which allows to describe decoherence due to the weak interaction with a Markovian bath and present the covariance matrix formalism, which allows the exact treatment of quadratic fermionic systems. In Sec. III we extend this formalism to decoherent systems with linear and Hermitian quadratic Lindblad operators. Then we come to the calculation of the steady states and the ADRs for relevant interesting examples in this framework in Secs. IV, V, and VI. Here we explicitly demonstrate the presence of dissipative phase transitions. In Sec. VII we propose a possible implementation with cold ions before concluding in Sec. VIII.

II. NOTATION AND METHODS

In this section we introduce our tools and notation, namely the Lindblad master equation and the fermionic covariance matrix formalism which is ideally suited for describing quasi-free fermionic systems (see Sec. II C).

A. Lindblad Master Equation

We treat the decoherence of a system in contact with a heat bath with the Lindblad master equation [30]

$$\partial_t \rho = \mathcal{S} \rho = -\frac{i}{\hbar} [\mathbf{H}, \rho] + \sum_{\alpha} \left(\mathbf{L}^{\alpha} \rho \mathbf{L}^{\alpha\dagger} - \frac{1}{2} \{ \mathbf{L}^{\alpha\dagger} \mathbf{L}^{\alpha}, \rho \} \right), \quad (3)$$

where ρ is the density matrix of the system, \mathbf{H} is its Hamiltonian, and the Lindblad operators \mathbf{L}^{α} determine the interaction between the system and the bath. This dynamical equation for an open system can be derived from two different points of view [31]: First, it can be derived from the full dynamics of system and bath. Here three major approximation have to be used: The states of system and environment are initially uncorrelated, the coupling between system and bath is weak (Born approximation), and the environment equilibrates fast (Markov approximation). Second, any time-evolution given by a quantum dynamical semigroup (i.e., a family of completely positive, trace preserving maps ϵ_t , which is strongly continuous and satisfies $\epsilon_t \epsilon_s = \epsilon_{t+s}$) is generated by a equation of the form Eq. 3.

We characterize the decoherence dynamics with the steady state and the asymptotic decay rate. A steady state density matrix ρ_0 of the master equation (3) fulfills

$$\partial_t \rho_0 = \mathcal{S} \rho_0 = 0 \quad (4)$$

and is the (generically unique) eigenvector with eigenvalue 0 of the Liouvillian superoperator \mathcal{S} . The approach to the steady state is then governed by the non-zero eigenvalues (and eigenvectors) of \mathcal{S} , all of which have non-positive real part for Liouvillians of Lindblad form. Of particular interest is the eigenvalue with the largest real part, since it governs the long-term dynamics. We refer to the absolute value of this largest real part as the *asymptotic decay rate* (ADR) and denote it by Δ :

$$\Delta(\mathcal{S}) = \max\{|\operatorname{Re} \lambda| \neq 0 : \exists \rho_{\lambda} : \mathcal{S}(\rho_{\lambda}) = \lambda \rho_{\lambda}\}. \quad (5)$$

B. Quasifree Fermions and Spins

We consider systems with N fermionic modes described by creation and annihilation operators a_j^{\dagger} and a_j . These operators obey the canonical anti-commutation relations

$$\{a_j, a_k\} = 0, \quad \{a_j^{\dagger}, a_k\} = \delta_{jk}. \quad (6)$$

Equivalently, we can use Hermitian fermionic Majorana operators

$$c_{j,0} = a_j^{\dagger} + a_j, \quad c_{j,1} = (-i)(a_j^{\dagger} - a_j), \quad (7)$$

which as generators of the Clifford algebra satisfy the anti-commutation relations

$$\{c_{j,u}, c_{k,v}\} = 2\delta_{jk}\delta_{uv}. \quad (8)$$

We consider fermionic Hamiltonians that are quadratic in the Majorana operators. They describe quasifree fermions and are known to be exactly solvable. We parameterize them with the real antisymmetric matrix H

$$\mathbf{H} = \frac{i}{4}\hbar \sum_{j,k,u,v} H_{jk,uv} c_{j,u} c_{k,v}. \quad (9)$$

The 2×2 matrix $H_{jk} \equiv (H_{jk,uv})_{uv}$ describes the coupling between the modes j and k .

All eigenstates and thermal states of such a quadratic fermionic Hamiltonian are Gaussian, i.e., they have a density operator which is the exponential of a quadratic form in the Majorana operators. Gaussian states remain Gaussian under the evolution with quadratic Hamiltonians.

In the following, we will mostly be concerned with *translationally invariant* systems and nearest-neighbour interactions. In terms of the matrix H the former means that H_{jk} depends only on the difference $j - k$ and we write for short

$$H_{jk} \equiv H_{j-k}, \quad (10)$$

while the latter implies that $H_s = 0$ for $s > 1$. We work with periodic boundary conditions, so $j - k$ is understood modulo N .

An important reason to study one-dimensional fermionic systems with quadratic Hamiltonian is their intimate relation to certain types of spin chains: The Jordan-Wigner transformation [32] maps fermionic operators onto Pauli spin operators via

$$c_{j,0} \leftrightarrow \prod_{k=1}^{j-1} \sigma_z^k \sigma_x^j, \quad c_{j,1} \leftrightarrow \prod_{k=1}^{j-1} \sigma_z^k \sigma_y^j. \quad (11)$$

Under this transformation some spin chains are mapped to spinless quasifree fermionic systems which can be solved exactly. A prominent example is the anisotropic XY chain in a transverse magnetic field [18] with the Hamiltonian

$$\begin{aligned} \mathbf{H} = & -J \sum_{j=1}^N [(1+\gamma)\sigma_x^j \sigma_x^{j+1} + (1-\gamma)\sigma_y^j \sigma_y^{j+1}] \\ & + B \sum_{j=1}^N \sigma_z^j, \end{aligned} \quad (12)$$

where B is the magnetic field, J the ferromagnetic coupling, and γ the anisotropy parameter. Closed systems governed by this Hamiltonian show a quantum phase transition at $B = 2J$ in the thermodynamic limit and

the behavior in the presence of dissipation is studied in Sec. VIB.

We are interested in dissipative (open) fermionic systems, with dynamics described by a Lindblad master equation, characterized by a set of Lindblad operators L^α . We consider two classes of Lindblad operators: firstly, those given by arbitrary linear combinations of the Majorana operators (*linear* Lindblad operators)

$$\mathbf{L}^\alpha = \sum_{j,u} L_{j,u}^\alpha c_{j,u}, \quad L_{j,u}^\alpha \in \mathbb{C}, \quad (13)$$

and secondly, those represented by quadratic expressions in the Majorana operators which are in addition Hermitian (*Hermitian quadratic* Lindblad operators)

$$\mathbf{L}^\alpha = \frac{i}{4} \sum_{j,k,u,v} L_{jk,uv}^\alpha c_{j,u} c_{k,v} \quad (14)$$

with the real and antisymmetric matrix L^α .

C. Covariance Matrix Formalism

Now we present a framework in which the dissipative dynamics of the Lindblad master equation (3) can be solved exactly.

For every state of a fermionic system, its real and antisymmetric covariance matrix (CM) is defined by

$$\Gamma_{jk,uv} = \text{tr} \left(\rho \frac{i}{2} [c_{j,u}, c_{k,v}] \right). \quad (15)$$

The magnitudes of the imaginary eigenvalues of Γ are smaller than or equal to unity ($\Gamma^2 \leq -\mathbb{1}$).

For Gaussian states the correlation functions of all orders are related to the covariance matrix through Wick's theorem [33]. In particular, pure Gaussian states $\rho = |\Psi\rangle\langle\Psi|$ satisfy $\Gamma^2 = -\mathbb{1}$. In our notation Γ_{jk} denotes a 2×2 matrix that describes the covariances between sites j and k .

III. LINDBLAD MASTER EQUATION IN THE COVARIANCE MATRIX FORMALISM

The covariance matrix formalism is especially useful if the operative dynamics leads to closed equations for the CM, which is the case for the two kinds of Lindblad operators Eqs. (13,14) that we study in the following.

A. Linear Lindblad operators

We consider a system with quadratic Hamiltonian given by the antisymmetric matrix H [cf. Eq. (9)] and linear Lindblad operators as defined in Eq. (13). Using

the anti-commutation relations (8) we determine the dynamical equation for the covariance matrix Γ from Eq. (3) and obtain:

$$\partial_t \Gamma = [H, \Gamma] - \sum_{\alpha} \{ |L^{\alpha}\rangle \langle L^{\alpha}| + |L^{\alpha*}\rangle \langle L^{\alpha*}|, \Gamma \} - 2i (|L^{\alpha}\rangle \langle L^{\alpha}| - |L^{\alpha*}\rangle \langle L^{\alpha*}|), \quad (16)$$

where $|L^{\alpha}\rangle$ denotes the vector formed by the coefficients $L_{j,u}^{\alpha}$ in Eq. (13) and $|L^{\alpha*}\rangle$ its complex conjugate. In terms of $|\Gamma\rangle$, the vector of components of Γ , this equation becomes

$$\partial_t |\Gamma\rangle = \mathcal{S}|\Gamma\rangle - |\mathcal{V}\rangle = (\mathcal{H} - \mathcal{M})|\Gamma\rangle - |\mathcal{V}\rangle, \quad (17)$$

with the superoperators

$$\mathcal{H} = (H \otimes \mathbf{1} - \mathbf{1} \otimes H^T), \quad (18)$$

$$\mathcal{M} = \sum_{\alpha} (|L^{\alpha}\rangle \langle L^{\alpha}| \otimes \mathbf{1} + \mathbf{1} \otimes (|L^{\alpha}\rangle \langle L^{\alpha}|)^T + \text{c.c.}), \quad (19)$$

$$|\mathcal{V}\rangle = 2i \sum_{\alpha} (|L^{\alpha}\rangle \otimes |L^{\alpha}\rangle - \text{c.c.}). \quad (20)$$

Note that \mathcal{H} is anti-Hermitian and \mathcal{M} is Hermitian and positive semi-definite. The steady state covariance matrix [see Eq. (4)] satisfies

$$(\mathcal{H} - \mathcal{M})|\Gamma_0\rangle = |\mathcal{V}\rangle. \quad (21)$$

Deviations $|\delta\Gamma\rangle = |\Gamma\rangle - |\Gamma_0\rangle$ then obey

$$\partial_t |\delta\Gamma\rangle = (\mathcal{H} - \mathcal{M})|\delta\Gamma\rangle \quad (22)$$

and the approach to the steady state is governed by the the right eigenvalues of the superoperator $\mathcal{S} = \mathcal{H} - \mathcal{M}$, satisfying

$$\mathcal{S}|\Gamma_i\rangle = \lambda_i |\Gamma_i\rangle. \quad (23)$$

The eigenvalues whose real parts are closest to zero thus determine the asymptotics of the decoherence process. In the following, we refer to

$$\Delta = \max \{ |\text{Re}\lambda_i| \neq 0 : \exists \Gamma_i \text{ s.t. } (\mathcal{S} - \lambda_i)|\Gamma_i\rangle = 0 \}, \quad (24)$$

i.e., the asymptotic decay rate on the level of covariance matrices simply as ADR.

B. Quadratic and Hermitian Lindblad operators

The second class of master equations leading to closed equations for the covariance matrix is of the form Eq. (3) with Lindblad operators that are quadratic and Hermitian, as in Eq. (14). Lindblad equations with Hermitian Lindblad operators describe the dynamics of systems in contact with a classical bath. Let us choose a fluctuating

external field as the source of decoherence (see Sec. VII). If, additionally, the Lindblad operators are quadratic, the fluctuating Hamiltonian is quadratic. Thus in this case Gaussian states evolve into mixtures of Gaussian states under such evolutions and we can expect a closed equation for the covariance matrix.

Before discussing the master equation in the CM formalism, let us first determine in general the steady state density matrices [see Eq. (4)] of a master equation with only Hermitian Lindblad operators. In that case, we can rewrite the master equation in terms of $|\rho\rangle$, the vector of components of ρ as

$$\partial_t |\rho\rangle = \mathcal{S}|\rho\rangle = \left(\mathcal{H} - \frac{1}{2} \sum_{\alpha} (\mathcal{L}^{\alpha})^2 \right) |\rho\rangle. \quad (25)$$

with the superoperators

$$\mathcal{H} = -i (\mathbf{H} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{H}^T), \quad (26)$$

$$\mathcal{L}^{\alpha} = \mathbf{L}^{\alpha} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{L}^{\alpha T}. \quad (27)$$

We observe that the superoperator \mathcal{H} is anti-Hermitian and that the superoperators \mathcal{L}^{α} are Hermitian, so that the $(\mathcal{L}^{\alpha})^2$ are Hermitian and non-negative.

We consider all complex valued vectors $|\rho\rangle$ instead of just the ones corresponding to positive density matrices with trace one. Therefore, we have to check after the calculation if our results correspond to physically meaningful states. The steady states satisfy

$$\langle \rho_0 | \left(\mathcal{H} - \frac{1}{2} \sum_{\alpha} (\mathcal{L}^{\alpha})^2 \right) | \rho_0 \rangle = 0. \quad (28)$$

As stated above, \mathcal{H} is anti-Hermitian and all $(\mathcal{L}^{\alpha})^2$ are Hermitian. Applying these properties we can conclude from Eq. (28) that

$$\langle \rho_0 | \sum_{\alpha} (\mathcal{L}^{\alpha})^2 | \rho_0 \rangle = \langle \rho_0 | \mathcal{H} | \rho_0 \rangle = 0 \quad (29)$$

holds. It follows from the non-negativity of $(\mathcal{L}^{\alpha})^2$ that

$$(\mathcal{L}^{\alpha})^2 | \rho_0 \rangle = 0 \quad \forall \alpha. \quad (30)$$

Because the \mathcal{L}^{α} can be diagonalized this implies $\mathcal{L}^{\alpha} | \rho_0 \rangle = 0$. It follows that $\mathcal{H} | \rho_0 \rangle$ vanishes identically. In terms of matrices ρ_0 , we can summarize these conditions for steady states

$$[\mathbf{H}, \rho_0] = [\mathbf{L}^{\alpha}, \rho_0] = 0 \quad \forall \alpha. \quad (31)$$

It can be verified with Eq. (3) that this condition for steady states is not only necessary but also sufficient. To summarize, steady states for Hermitian Lindblad operators correspond to density matrices commuting with the Hamiltonian and all Lindblad operators. Therefore, they are the identity up to symmetries shared by the Hamil-

tonian and the Lindblad operators.

Let us now return to exactly solvable systems in the CM formalism. For quadratic and Hermitian Lindblad operators and quadratic Hamiltonians the Master Equation (3) becomes

$$\partial_t \Gamma = [H, \Gamma] + \frac{1}{2} \sum_{\alpha} [L^{\alpha}, [L^{\alpha}, \Gamma]]. \quad (32)$$

We can again reformulate this equation for the vector of components $|\Gamma\rangle$

$$\partial_t |\Gamma\rangle = S |\Gamma\rangle = \left(\mathcal{H} - \frac{1}{2} \sum_{\alpha} (\mathcal{L}^{\alpha})^2 \right) |\Gamma\rangle, \quad (33)$$

with \mathcal{H} as in Eq. (18) and $\mathcal{L}^{\alpha} = L^{\alpha} \otimes \mathbb{1} - \mathbb{1} \otimes L^{\alpha}$.

Since we found that steady states are trivial for Hermitian Lindblad operators, we concentrate on the asymptotics of the decoherence process. It is studied through the eigenvalues λ_i of the superoperator \mathcal{S} , and in particular its ADR as defined in Eq. (24).

C. Translationally invariant Hamiltonians

Naturally, translationally invariant systems are best treated in a Fourier transformed picture. Any real antisymmetric matrix can be transformed into a real and antisymmetric block-diagonal matrix by an orthogonal transformation O . For the Hamiltonian matrix H this means

$$H'_{mn,uv} = (OHO^T)_{mn,uv}, \quad H'_{mn} = \delta_{mn} \begin{pmatrix} 0 & \epsilon_m \\ -\epsilon_m & 0 \end{pmatrix}, \quad (34)$$

where the real number ϵ_m are the energies of the elementary excitations. We, however, transform the Hamiltonian matrix with the unitary Fourier transform

$$\tilde{H}_{mn,uv} = (UHU^{\dagger})_{mn,uv}, \quad U_{mn,uv} = \frac{1}{\sqrt{N}} e^{\frac{2\pi i}{N} mn} \delta_{uv}. \quad (35)$$

The resulting matrix \tilde{H} is anti-Hermitian, but not real. For translationally invariant systems, for which the 2×2 matrices H_{jk} in Eq. (9) depend only on $j - k$, the matrix \tilde{H} is block-diagonal with

$$\tilde{H}_{mn} = \delta_{mn} \sum_{s=0}^{N-1} H_s e^{-\frac{2\pi i}{N} sm}. \quad (36)$$

The block-diagonal is parameterized according to

$$\tilde{H}_{nn} = \begin{pmatrix} ik_n & h_n \\ -h_n^* & il_n \end{pmatrix}, \quad k_n, l_n \in \mathbb{R}, \quad h_n \in \mathbb{C}. \quad (37)$$

For later use, we observe the properties

$$h_{-n} = h_n^*, \quad k_{-n} = -k_n, \quad l_{-n} = -l_n, \quad (38)$$

which follow directly from Eq. (36) for real H_s .

For a system that is also invariant under reflections (in real space) $H_s = -H_s^T$ holds (in addition to $H_{-s} = -H_s^T$ implied by antisymmetry). In that case, we have $\tilde{H}_{nn} = -\tilde{H}_{nn}$ and therefore

$$k_n = l_n = 0. \quad (39)$$

The spectrum of the Hamiltonian matrix determines the elementary excitation energies

$$\epsilon_n = \left| \frac{k_n + l_n}{2} \pm \sqrt{\left(\frac{k_n - l_n}{2} \right)^2 + |h_n|^2} \right|. \quad (40)$$

It will be necessary to transform the covariance matrix Γ accordingly, defining

$$\tilde{\Gamma} = U \Gamma U^{\dagger}. \quad (41)$$

By minimizing the energy expectation value

$$\langle E \rangle = \text{Tr}(H^T \Gamma) = \text{Tr}(\tilde{H}^{\dagger} \tilde{\Gamma}), \quad (42)$$

we find the covariance matrix for the ground state. In the case $k_n l_n < |h_n|^2$ it is

$$\tilde{\Gamma}_{mn}^0 = \delta_{mn} \left[\left(\frac{k_n - l_n}{2} \right)^2 + |h_n|^2 \right]^{-1/2} \begin{pmatrix} i \frac{k_n - l_n}{2} & -h_n \\ h_n^* & -i \frac{k_n - l_n}{2} \end{pmatrix} \quad (43)$$

and otherwise

$$\tilde{\Gamma}_{mn}^0 = -i \delta_{mn} \text{sign}(k_n + l_n) \mathbb{1}_2. \quad (44)$$

For translationally invariant and reflection symmetric systems $k_n l_n = 0$ holds, thus $k_n l_n < |h_n|^2$ is fulfilled in such systems. Since the XY chain Eq. 12 is reflection symmetric, we can concentrate on the case of Eq. (43). Specifically, we obtain for the Hamiltonian Eq. 12 that

$$h_n = -2B + 2J \left[(1 + \gamma) e^{\frac{2\pi i}{N} n} + (1 - \gamma) e^{-\frac{2\pi i}{N} n} \right], \quad (45)$$

$$k_n = l_n = 0, \quad (46)$$

which contains a continuous quantum phase transition at $B = 2J$, where the gap closes and an elementary excitation energy $\epsilon_n = |h_n| = 0$ exists. This Hamiltonian will be further discussed in Sec. VI.

IV. LINEAR LINDBLAD OPERATORS

Now we apply the formalism introduced in the previous Sections to some simple cases of physical interest. Here we choose the simplest examples, i.e., linear Lindblad operators (see Sec. III A). We study two settings. In Sec. IV A we look at systems without any unitary evolution, observing dynamic transitions when tuning the strength of competing decoherence processes. Here we

enrich our presentation with an example for dissipative state engineering. In Sec. IV B we consider open systems governed by a Hamiltonian, which describes a quantum phase transition itself, and show that the dissipative system undergoes a transition for the same values of the system parameters.

A. Purely dissipative systems

The simplest example of two competing decoherence processes generated by linear Lindblad operators is

$$\mathbf{L}_-^\alpha = g\mu a_\alpha, \quad \mathbf{L}_+^\alpha = g\nu a_\alpha^\dagger, \quad (47)$$

acting on site $\alpha \in \{1, \dots, N\}$. It describes the competition between particle-loss and particle-gain processes. We observe that the Master Equation (16) without the Hamiltonian ($H = 0$) is diagonal in real space

$$\partial_t \Gamma = -g^2(\mu^2 + \nu^2)\Gamma - g^2(\mu^2 - \nu^2) \bigoplus_{\alpha=1}^N (i\sigma_y). \quad (48)$$

In this simple case the master equation is already diagonal and we read off the single decoherence rate $\Delta = g^2(\mu^2 + \nu^2)$. Solving the master equation for $\partial_t \Gamma_0 = 0$ gives the unique steady state covariance matrix

$$\Gamma_0 = -\frac{\mu^2 - \nu^2}{\mu^2 + \nu^2} \bigoplus_{\alpha=1}^N \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (49)$$

which is block diagonal. This state is characterized by the particle number $\langle a_\alpha^\dagger a_\alpha \rangle = \nu^2/(\mu^2 + \nu^2)$ at all sites. For pure particle-loss processes ($\nu = 0$), all sites are unoccupied $\langle a_\alpha^\dagger a_\alpha \rangle = 0$ in the steady state, while for pure particle-gain processes ($\mu = 0$), all sites are occupied $\langle a_\alpha^\dagger a_\alpha \rangle = 1$. At $\mu = \nu$ the steady state is the unpolarized completely mixed state. Not surprisingly, the system does not display any phase transition.

More interesting may be the case in which dissipation can also induce correlations. A simple example of this kind is provided by the Lindblad operators

$$\mathbf{L}^\alpha = g \left(\mu a_\alpha + \nu a_{\alpha+1}^\dagger \right) \quad (50)$$

acting on nearest neighbors. This set of Lindblad operators generates a master equation, which is diagonal after

the Fourier transform (35)

$$\begin{aligned} \partial_t \tilde{\Gamma} = & -g^2(\mu^2 + \nu^2)\tilde{\Gamma} \\ & -g^2\mu\nu \left\{ \bigoplus_{n=1}^N \cos(2\pi n/N) \sigma_z, \tilde{\Gamma} \right\} \\ & -g^2(\mu^2 - \nu^2) \bigoplus_{n=1}^N i\sigma_y \\ & -2g^2\mu\nu \bigoplus_{n=1}^N i \sin(2\pi n/N) \sigma_x. \end{aligned} \quad (51)$$

In this case, a spectrum of decoherence rates $g^2\{\mu^2 + \nu^2 \pm 2\mu\nu[\cos \frac{2\pi n}{N} + \cos \frac{2\pi m}{N}], \mu^2 + \nu^2 \pm 2\mu\nu[\cos \frac{2\pi n}{N} - \cos \frac{2\pi m}{N}]\}$ exists with a “gap” $g^2(\mu - \nu)^2$. The unique steady state is

$$\tilde{\Gamma}_0 = -\frac{\mu^2 - \nu^2}{\mu^2 + \nu^2} \bigoplus_{n=1}^N i\sigma_y - \frac{2\mu\nu}{\mu^2 + \nu^2} \bigoplus_{n=1}^N i \sin(2\pi n/N) \sigma_x. \quad (52)$$

This state is a paired fermionic state according to the definition of Kraus *et al.* [34]. Paired states show two-particle quantum correlations that can not be reproduced by separable states (mixtures of Slater determinants). It is proven in [34] that Gaussian states are paired iff $Q_{kl} = \langle \frac{i}{2}[a_k, a_l] \rangle \neq 0$. This condition expresses the fact that separable states are convex combinations of states with a fixed particle number. For the covariance matrix (52) we get

$$Q_{kl} = \begin{cases} \frac{1}{2} \frac{\mu\nu \cdot \text{sign}(k-l)}{\mu^2 + \nu^2} & \text{if } |k-l| = 1 \\ 0 & \text{if } |k-l| \neq 1 \end{cases}. \quad (53)$$

We conclude that (50) generates paired states, except for the trivial cases $\mu = 0$ or $\nu = 0$. Note that even though the gap closes at $\mu = \nu$ (where maximal pairing is created) there is no phase transition at this point.

B. Dissipative systems with Hamiltonians

A different form of transitions can arise in the presence of a Hamiltonian when tuning the parameters of the Hamiltonian. To show this, we solve the evolution of the Lindblad master equation (16) with a general quadratic and translationally invariant Hamiltonian [see Eqs. (9) and (37)]. We choose the local Lindblad operators (47), again because they are the simplest example. The diag-

onal master equation in Fourier space becomes

$$\partial_t \tilde{\Gamma} = \left[\bigoplus_{n=1}^N \begin{pmatrix} ik_n & h_n \\ -h_n^* & il_n \end{pmatrix}, \tilde{\Gamma} \right] - g^2(\mu^2 + \nu^2) \tilde{\Gamma} - g^2(\mu^2 - \nu^2) \bigoplus_{n=1}^N i\sigma_y. \quad (54)$$

The corresponding steady state covariance matrix in the weak-coupling limit $g \rightarrow 0$ is [35]

$$\tilde{\Gamma}_0 = -\frac{\mu^2 - \nu^2}{\mu^2 + \nu^2} \bigoplus_{n=1}^N \frac{\text{Re}(h_n)}{(k_n - l_n)^2/4 + |h_n|^2} \cdot \begin{pmatrix} i(k_n - l_n)/2 & h_n \\ -h_n^* & -i(k_n - l_n)/2 \end{pmatrix}. \quad (55)$$

Transforming back to Γ_0 [and using Eq. (38)] we can read off the particle number $\langle 2a_j^\dagger a_j - 1 \rangle = (\Gamma_0)_{jj,01}$ as

$$(\Gamma_0)_{jj,01} = \frac{1}{2} \frac{\mu^2 - \nu^2}{\mu^2 + \nu^2} \frac{1}{N} \sum_{n=1}^N \frac{\text{Re}(h_n)^2}{(k_n - l_n)^2/4 + |h_n|^2}. \quad (56)$$

Based on this result we can now discuss how non-analytic behavior in the steady state correlates with critical points of the system. A vanishing denominator in Eq. (56) is not a priori a sufficient condition for non-analytic behavior because the numerator might vanish at the same point. This is relevant for interesting examples with $k_n - l_n = 0$, e.g., the XY chain in Eq. (45). We give a rigorous discussion in the following. In the thermodynamic limit, the sums over expectation values in Eq. (56) can be replaced by a loop integral around the origin of the complex plane with radius one, where the integration variable is $z = \exp(\frac{2\pi i}{N}n)$. This is possible because h_n , k_n , and l_n are Fourier series. For local interactions, the denominator of the integrand is a polynomial in z [see Eq. (35)] and thus has a finite number of distinct roots. Applying the residue theorem, a non-analyticity in $\langle a_j^\dagger a_j \rangle$ is possible only if a residue of the integrand, i.e., a root of its denominator, moves through the integral contour in the complex plane as a function of some external parameters. This happens for a vanishing denominator $|h_n|^2 + (k_n - l_n)^2/4 = 0$ for some real $n \in [0, N)$. In the special case of a reflection symmetric system $k_n + l_n = 0$ this coincides with a vanishing energy gap $\epsilon_n = 0$ [see Eq. (40)], a signature for a quantum phase transition. To summarize, for a reflection symmetric system with $|h_n|^2 + (k_n - l_n)^2/4 = 0$ in the weak-coupling limit a quantum phase transition occurs in the dissipative system for the same parameter values as in the corresponding closed system and is signaled by a non-analyticity in $\langle a_j^\dagger a_j \rangle$. This calculation is explicitly performed in section VIA for the XY chain [36].

V. QUADRATIC AND HERMITIAN LINDBLAD OPERATORS

In this Section we turn to the dynamical properties of the Lindblad master equation with quadratic and Hermitian Lindblad operators as introduced in Sec. IIIB. In the study of closed systems, quantum phase transitions are signalled by non-analyticities in ground state expectation values. In the dissipative case the steady state is the analog of the ground state. However, we have shown in Sec. IIIB that in the case of Hermitian Lindblad operators the steady states are trivial and thus cannot evidence a phase transition. Therefore we turn to the ADR, which determines the long-time dynamics of the decoherence process. We identify non-analytical behavior of this rate both in the absence of any Hamiltonian (see Sec. VA) for competing decoherence processes and for non-zero Hamiltonian, in which case phase transitions of the corresponding closed system are reflected in a “dynamical transition” of this rate (see Sec. VB).

A. Purely dissipative systems

A particular simple set of local and quadratic Lindblad operators is

$$\mathbf{L}_z^\alpha = g\mu \frac{i}{2} [c_{\alpha,1}, c_{\alpha,0}], \quad (57)$$

$$\mathbf{L}_x^\alpha = g\nu \frac{i}{2} [c_{\alpha+1,0}, c_{\alpha,1}]. \quad (58)$$

In this case the Lindblad equation (32) becomes

$$\begin{aligned} \partial_t \Gamma_{kl,uv} = & -4g^2\mu^2 \Gamma_{kl,uv} (1 - \delta_{kl}) \\ & -4g^2\nu^2 \Gamma_{kl,uv} (1 - \delta_{2k+u+1, 2l+v} \delta_{k+1, l} \\ & \quad - \delta_{2k+u-1, 2l+v} \delta_{k-1, l}), \end{aligned} \quad (59)$$

We can read off the decoherence rates $-4g^2(\mu^2 + \nu^2)$, $-4g^2\mu^2$, and $-4g^2\nu^2$. Thus, the ADR

$$\Delta = \begin{cases} 4g^2\mu^2 & \text{if } \mu \leq \nu \\ 4g^2\nu^2 & \text{if } \nu < \mu \end{cases} \quad (60)$$

undergoes a dynamical transition as a function of μ/ν at $\mu = \nu$.

B. Dissipative systems with Hamiltonian

Now we add a quadratic Hamiltonian and calculate the asymptotic decay rate Δ in the limit of small couplings to the environment $g \rightarrow 0$. First, we derive it for the quadratic Lindblad operators from Eqs. (57,58) for $\nu = 0$ and $\mu = 1$. Later we will present the results for the case of arbitrary μ and ν . For translationally invariant systems

the Fourier transformed master equation (59) is

$$\partial_t \tilde{\Gamma}_{kl} \equiv (\tilde{\mathcal{S}}\tilde{\Gamma})_{kl} = [\tilde{H}, \tilde{\Gamma}]_{kl} - 4g^2 \left(\tilde{\Gamma}_{kl} - \frac{1}{N} \sum_{r,s=1}^N \tilde{\Gamma}_{rs} \delta_{r-s,k-l} \right), \quad (61)$$

with the unitarily transformed superoperator $\tilde{\mathcal{S}}$ to

$$\tilde{\mathcal{S}} = (U \otimes U) \mathcal{S} (U \otimes U)^\dagger, \quad (62)$$

with U from Eq. (35). For weak couplings between system and bath $g \rightarrow 0$, the eigenvalues of $\tilde{\mathcal{S}}$ (and thus of \mathcal{S}) can be determined by first order perturbation expansion. To this end we first diagonalize the unperturbed Hamiltonian part of $\tilde{\mathcal{S}}$

$$[\tilde{H}, \tilde{\Gamma}]_{kl} = \tilde{H}_{kk} \tilde{\Gamma}_{kl} - \tilde{\Gamma}_{kl} \tilde{H}_{ll} \stackrel{!}{=} \lambda \tilde{\Gamma}_{kl}, \quad (63)$$

where we use the notation introduced in Eq. (36) for the Hamiltonian \tilde{H} . The $4N^2$ eigenvalues λ^{mna} ($m, n = 1, \dots, N, a = 1, \dots, 4$) are

$$\lambda^{mn1} = i(\alpha_m - \alpha_n + \beta_m - \beta_n), \quad (64)$$

$$\lambda^{mn2} = i(\alpha_m - \alpha_n - \beta_m + \beta_n), \quad (65)$$

$$\lambda^{mn3} = i(\alpha_m - \alpha_n + \beta_m + \beta_n), \quad (66)$$

$$\lambda^{mn4} = i(\alpha_m - \alpha_n - \beta_m - \beta_n), \quad (67)$$

with

$$\alpha_m = |k_m + l_m|/2, \quad (68)$$

$$\beta_m = \sqrt{|h_m|^2 + (k_m - l_m)^2/4}. \quad (69)$$

The corresponding eigenmatrices are denoted as $\tilde{\Lambda}^{mna}$ with nonzero elements $\tilde{\Lambda}_{kl}^{mna}$ only for $m = k$ and $n = l$, i.e., $\tilde{\Lambda}_{kl}^{mna} = \delta_{mk} \delta_{nl} \tilde{\Lambda}_{mn}^{mna}$. Perturbation theory demands to calculate the matrix elements of the perturbative part of $\tilde{\mathcal{S}}$, $-4g^2(\delta_{mk} \delta_{nl} \delta_{ab} - P_{klb}^{mna}/N)$ [see Eq. (61)], with

$$\begin{aligned} P_{klb}^{mna} &= \frac{N}{4g^2} \langle \tilde{\Lambda}^{mna} | \frac{1}{2} \sum_{\alpha} (\mathcal{L}^{\alpha})^2 | \tilde{\Lambda}^{klb} \rangle + N \delta_{mk} \delta_{nl} \delta_{ab}, \\ &= \sum_{q,r,s,t=1}^N \delta_{q-r,s-t} \text{Tr} \left[(\tilde{\Lambda}_{st}^{mna})^\dagger \tilde{\Lambda}_{qr}^{klb} \right], \\ &= \delta_{m-n,k-l} \text{Tr} \left[(\tilde{\Lambda}_{mn}^{mna})^\dagger \tilde{\Lambda}_{kl}^{klb} \right]. \end{aligned} \quad (70)$$

Thus the eigenvalues of $\tilde{\mathcal{S}}$ are determined by those of the Hermitian matrix P and the largest eigenvalue smaller than N of P (restricted to a space of degenerate eigenvalues λ^{mna} of $[\tilde{H}, \cdot]$) determines the ADR. We denote it by Δ_P and thus have that the asymptotic rate is $\Delta = 4g^2(1 - \frac{\Delta_P}{N})$. To find Δ_P , note that the matrix elements of P fulfill $|P_{klb}^{mna}| \leq 1$. Thus an N -fold degeneracy of λ^{mna} is required for $\Delta_P = \Omega(N)$. Generically,

this is possible only for the eigenvalue $\lambda^{mna} = 0$, i.e., $m = n$ and $a = 1, 2$. The corresponding eigenmatrices are

$$\tilde{\Lambda}_{kl}^{mm1} = \delta_{mk} \delta_{ml} \frac{1}{\sqrt{2}\beta_n} \begin{pmatrix} i \frac{k_m - l_m}{2} & -h_m \\ h_m^* & -i \frac{k_m - l_m}{2} \end{pmatrix}, \quad (71)$$

$$\tilde{\Lambda}_{kl}^{mm2} = \delta_{mk} \delta_{ml} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (72)$$

As the eigenmatrices $\tilde{\Lambda}^{mm2}$ give eigenvalues equal to N and 0 only, have no overlap with physical covariance matrices, and yield $P_{kl1}^{mm2} = 0$, we focus on the matrices $\tilde{\Lambda}^{mm1}$. The corresponding part of the perturbation matrix is

$$P_{mn} = P_{nn1}^{mm1} = \frac{2h_m h_n^* + 2h_m^* h_n - (k_m - l_m)(k_n - l_n)}{4\beta_m \beta_n}. \quad (73)$$

We diagonalize this matrix by introducing the three vectors $|a\rangle, |b\rangle, |c\rangle \in \mathbb{C}^N$ with the components

$$a_m = \frac{k_m - l_m}{2\beta_m}, \quad b_m = \frac{\text{Im}(h_m)}{\beta_m}, \quad c_m = \frac{\text{Re}(h_m)}{\beta_m}, \quad (74)$$

and writing P_{mn} in terms of these unnormalized vectors

$$P = |c\rangle\langle c| + |b\rangle\langle b| - |a\rangle\langle a|. \quad (75)$$

We now exploit the symmetries of h_n, k_n , and l_n stated in Eq. (38). First we observe that $|c\rangle$ is orthogonal to $|a\rangle$ and $|b\rangle$. We have chosen the covariance matrices corresponding to the three vectors (74) anti-Hermitian, since this matrix remains anti-Hermitian even in the complex vector space. After transforming back into real space the ones corresponding to $|a\rangle$ and $|b\rangle$ are purely imaginary so that they have no overlap with any physically meaningful real and antisymmetric covariance matrix. Only the matrix corresponding to $|c\rangle$ is real and antisymmetric and given by

$$\Gamma_{\Delta} = \left(\sum_{m=1}^N \frac{|h_m|^2}{2\beta_m^2} \right)^{-1} \sum_n \frac{\text{Re}(h_n)}{\beta_n} U^\dagger \Lambda^{nn1} U. \quad (76)$$

Therefore, it determines the ADR. We get

$$\Delta_P = \sum_{m=1}^N \frac{\text{Re}(h_m)^2}{\beta_m^2} = \sum_{m=1}^N \frac{\text{Re}(h_m)^2}{|h_m|^2 + (k_m - l_m)^2/4}, \quad (77)$$

and thus

$$\Delta = \frac{4g^2}{N} \sum_{m=1}^N \frac{4\text{Im}(h_m)^2 + (k_m - l_m)^2}{4|h_m|^2 + (k_m - l_m)^2} \quad (78)$$

as the general form of the ADR.

We can extend our analysis to systems with the general Lindblad operators Eqs. (57,58) and find in an analog

way the two lowest decay rates

$$\frac{\Delta_{\pm}}{4g^2} = \mu^2 + \nu^2 - \frac{\epsilon_z + \epsilon_x}{2} \pm \sqrt{\left(\frac{\epsilon_z - \epsilon_x}{2}\right)^2 + \epsilon^2}, \quad (79)$$

with

$$\begin{aligned} \epsilon_z &= \frac{\mu^2}{N} \sum_{m=1}^N \frac{\text{Re}(h_m)^2}{\beta_m^2}, \\ \epsilon_x &= \frac{\nu^2}{N} \sum_{m=1}^N \frac{\text{Re}(h_m \exp(-2\pi im/N))^2}{\beta_m^2}, \\ \epsilon &= \frac{\mu\nu}{N} \sum_{m=1}^N \frac{\text{Re}(h_m) \text{Re}(h_m \exp(-2\pi im/N))}{\beta_m^2}. \end{aligned} \quad (80)$$

We can now argue that the ADR itself reflects the criticality of the system. The argument is completely analogous to the one given in Sec. IV B. If the denominator becomes zero, we can expect a non-analyticity expressions Eqs. (80). In particular, in the reflection symmetric case $k_n + l_n = 0$, where the denominator agrees with the elementary excitation energies [$\epsilon_n^2 = |h_n|^2 + (k_n - l_n)^2/4 = 0$, see Eq. (40)] the non-analyticity in the ADR signals the presence of a quantum phase transition in the Hamiltonian itself.

VI. EXAMPLE HAMILTONIANS

In this section we will revisit the results obtained for the steady state and the ADR for linear and quadratic Lindblad operators in Secs. IV B and V B for the specific Hamiltonian (12) of the quantum XY chain.

The energies of the elementary excitations of this Hamiltonian are $\epsilon_n = |h_n|$. Thus, for the XY chain in Eq. (45) the gap closes at $B = 2J$ in the thermodynamic limit and the quantum XY chains exhibit a phase transition at this point. In fact, these models constitute the archetypal example of a continuous quantum phase transition [18]. In this chapter we want to find properties of the dissipative dynamics signaling this phase transition.

A. Linear Lindblad operators

Let us now apply the findings from Sec. IV B and Eq. (56) to the example system defined in Eq. (45) which contains a quantum phase transition at $B = 2J$. Then the particle numbers become for $k_n = l_n = 0$

$$\langle 2a_n^\dagger a_n - 1 \rangle = \frac{1}{2} \frac{\mu^2 - \nu^2}{\mu^2 + \nu^2} \left(1 + \frac{1}{N} \sum_{n=1}^N \frac{h_n^*}{h_n} \right). \quad (81)$$

For $\gamma = 0$ we easily obtain $\Delta = 0$ (since by Eq. (45) h_n is real in that case and then by Eq. (78) Δ is zero for $k_n = l_n = 0$). For $\gamma \neq 0$ we evaluate the sum

$1/N \cdot \sum_{m=1}^N h_m^*/h_m$ in the thermodynamic limit by introducing the complex variable $z = \exp(-2\pi im/N)$

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{m=1}^N \frac{h_m^*}{h_m} &= \\ \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} \frac{2J(1-\gamma)z^2 - 2Bz + 2J(1+\gamma)}{2J(1+\gamma)z^2 - 2Bz + 2J(1-\gamma)}, \end{aligned} \quad (82)$$

where the integration contour is a circle of radius $|z| = 1$ around $z = 0$ in the complex plane. The complex integrand is analytic except for three distinct poles at

$$\begin{aligned} z^0 &= 0, \\ z^{\pm} &= \frac{1}{2J(1+\gamma)} \left[B \pm \sqrt{B^2 - 4J^2(1-\gamma^2)} \right]. \end{aligned} \quad (83)$$

The contour integral is determined by the sum over the

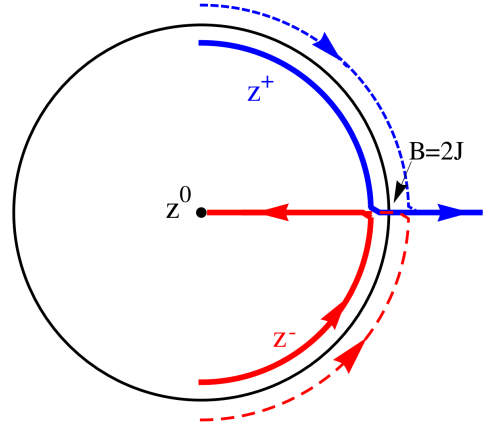


FIG. 1. The poles z^0, z^{\pm} [see Eq. (83)] are plotted for $J = 1, \gamma = \pm 0.1$. As B is changed from 0 to 20 the poles $z^+(z^-)$ for positive anisotropy $\gamma = +0.1$ move along the blue (red) solid curves and z^+ crosses the contour at the critical value $B = 2J$. For negative $\gamma = -0.1$, z^- crosses at $B = 2J$. At the crossing the integral Eq. (82) changes non-analytically.

residues at those poles which are inside the contour ($|z| < 1$). z^0 is always inside this contour. In the case $\gamma > 0$, z^+ is inside the contour for $0 \leq B < 2J$ and outside for $B > 2J$, while z^- is always inside the contour. In the case $\gamma < 0$, z^- is inside the contour for $B > 2J$ and outside for $0 \leq B < 2J$, while z^+ is always outside the contour. So residues cross the contour at the quantum phase transition $B = 2J$ (because then $h_n = 0$ for some n), leading to a non-analytical behavior in the particle density of the steady state.

After applying the residue theorem we get the particle

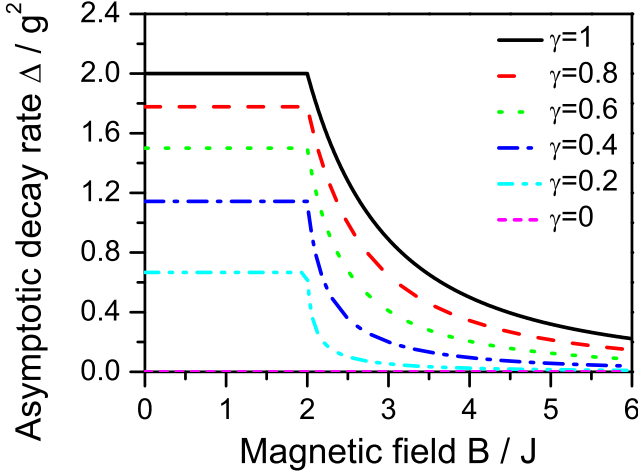


FIG. 2. Asymptotic decay rate Δ [see Eq. (78)] of the XY chain (12) for different anisotropy parameters γ as a function of the magnetic field in the limits $N \rightarrow \infty$ and $g \rightarrow 0$. A phase transition in Δ is visible at $B = 2J$ for $\gamma \neq 0$.

number of the steady state

$$\langle 2a_n^\dagger a_n - 1 \rangle = \begin{cases} \frac{\mu^2 - \nu^2}{\mu^2 + \nu^2} & B \leq 2J \\ \frac{1}{1-\gamma^2} \left(1 - \frac{\gamma^2}{\sqrt{1 - (\frac{2J}{B})^2 (1-\gamma^2)}} \right) & B \geq 2J \end{cases} \quad (84)$$

for all γ , which does not depend on the sign of γ . For $B < 2J$ the particle number in the steady state does not vary with the magnetic field, while its magnitude approaches $(\mu^2 - \nu^2)/(\mu^2 + \nu^2)$ for large magnetic fields like $\sim (J/B)^2$. To summarize, the steady state undergoes a dissipative phase transition at $B = 2J$ signaling the phase transition in the system.

B. Quadratic and Hermitian Lindblad operators

As an example we study the anisotropic XY chain in a transverse magnetic field with the Hamiltonian given in Eq. (12). This translationally invariant Hamiltonian is Jordan-Wigner transformed to a quadratic fermionic Hamiltonian with Hamiltonian matrix H given by

$$H_0 = \begin{pmatrix} 0 & -2B \\ 2B & 0 \end{pmatrix}, \quad (85)$$

$$H_1 = \begin{pmatrix} 0 & 2J(1-\gamma) \\ -2J(1+\gamma) & 0 \end{pmatrix}, \quad (86)$$

$$H_{-1} = \begin{pmatrix} 0 & 2J(1+\gamma) \\ -2J(1-\gamma) & 0 \end{pmatrix}. \quad (87)$$

After Fourier transforming [see Eq. (35)] this Hamiltonian matrix assumes the form given in Eq. (37) with parameters h_n, k_n, l_n given by (45).

We now apply the results from Sec. VB to the Hamiltonian Eq. (85) and the Lindblad operators

$$\mathbf{L}^\alpha = g\mu \frac{i}{2} [c_{\alpha,1}, c_{\alpha,0}] \leftrightarrow g\sigma_z^\alpha. \quad (88)$$

After a brief discussion of the steady states and a derivation of the ADR in the thermodynamic ($N \rightarrow \infty$) and in weak coupling ($g \rightarrow 0$) limits, we present numerical results of the system dynamics for finite N and g and compare them with our analytic predictions.

First, we discuss the steady states of these systems (see Sec. IIIB). From Eq. (31) we have concluded that the steady state density matrix is the identity up to symmetries shared by the Lindblad operators and the Hamiltonian. A rigorous derivation of the steady states for this example could start from the ansatz that the steady state density matrix is diagonal in the Fock basis, following from $[\sigma_z^\alpha, \rho] = 0$. Then the commutator $[\mathbf{H}, \rho] = 0$ must be exploited to get the steady state.

As the Lindblad operators correspond to local particle number operators, the important compatible symmetries for the XY chains are the parity $\mathcal{P} = \sigma_z^1 \dots \sigma_z^N$, discriminating between an odd and an even number of particles, and the total particle number $\mathcal{N} = (\mathbf{1} + \sum \sigma_z^j)/2$. For truly asymmetric XY chains $\gamma \neq 0.5$ the parity is the highest symmetry compatible with the Lindblad operators. In these cases the steady state density matrix is given by the identity in the two sectors of even and odd parity, the relative weight of these sectors is determined by the initial state. For the symmetric chain $\gamma = 0.5$, the steady state density matrix is the identity only in the sectors with a constant total number of particles. Thus for $\gamma \neq 0.5$ the steady state magnetization is $\langle \sigma_z^j \rangle = 0$ regardless of the initial state, whereas the magnetization of the initial state is conserved for $\gamma = 0.5$.

Second, we calculate the ADR (78) for the XY chains with Eq. (45) analogous to the integration in Sec. VIA.

After applying the residue theorem we get the ADR

$$\Delta = 4g^2 \begin{cases} \frac{|\gamma|}{1+|\gamma|} & B \leq 2J \\ \frac{\gamma^2}{1-\gamma^2} \left(\left[1 - \left(\frac{2J}{B} \right)^2 (1-\gamma^2) \right]^{-1/2} - 1 \right) & B \geq 2J \end{cases} \quad (89)$$

for all γ in the case $\mu = 1$ and $\nu = 0$. It does not depend on the sign of γ and is shown in Fig. 2 for several values of $\gamma \in [0, 1]$. For $B < 2J$ the ADR does not vary with the magnetic field, while its magnitude decays to zero for large magnetic fields like $\sim (J/B)^2$. The same behavior was found for the variance of the particle number in these models in a previous work [37]. To summarize, the ADR undergoes a dissipative phase transition at $B = 2J$ signaling the phase transition in the system.

The final result for the ADR (89) is valid in the limits $N \rightarrow \infty$ and $g \rightarrow 0$. In this section we perform a

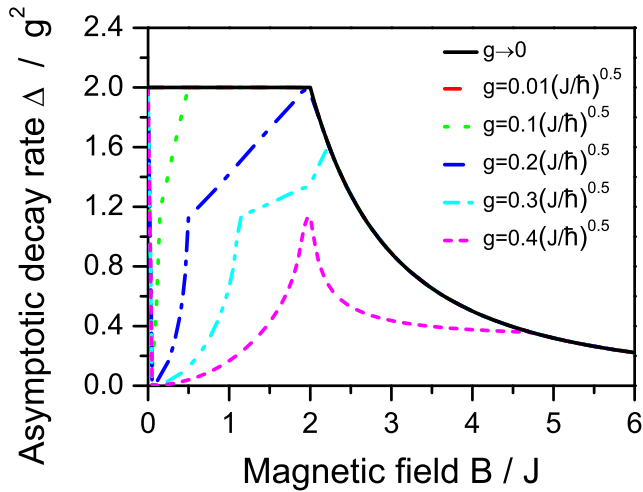


FIG. 3. Asymptotic decay rate Δ [see Eq. (78)] of the XY chain (12) for different coupling strengths g , $\gamma = 1$, and $N = 100$ as a function of the magnetic field B . For $g \leq 0.1 (J/\hbar)^{0.5}$ the results agree with the limit of weak coupling $g \rightarrow 0$ [see Eq. (89)].

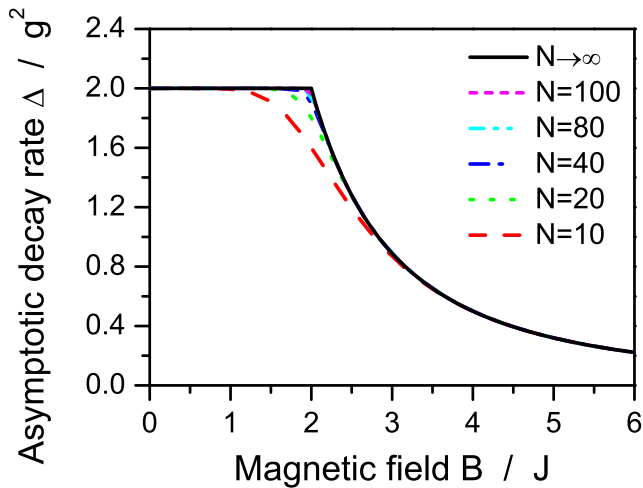


FIG. 4. Asymptotic decay rate Δ [see Eq. (89)] of the XY chain (12) for different system sizes N and $\gamma = 1$, $g = 0.01 (J/\hbar)^{0.5}$ as a function of the magnetic field B . For $N \geq 50$ the thermodynamic limit is reached except for small variations at the phase transition $B = 2J$.

numerical diagonalization of the Lindblad master equation superoperator \mathcal{S} to compare the analytic result with the values for finite N and g . Furthermore, we extract the ADR from a simulation of the system dynamics and compare it with our prediction.

In Fig. 3 we present the ADR for finite coupling strengths g . For $g^2 \leq 0.01 J/\hbar$ the result of perturbation theory is in excellent agreement with the numerical diagonalization of the Lindblad master equation superoperator. Deviations are strongest at small magnetic fields. We show the ADR Δ for different finite system sizes in Fig. 4. Even in small systems with $N = 10$ spins the

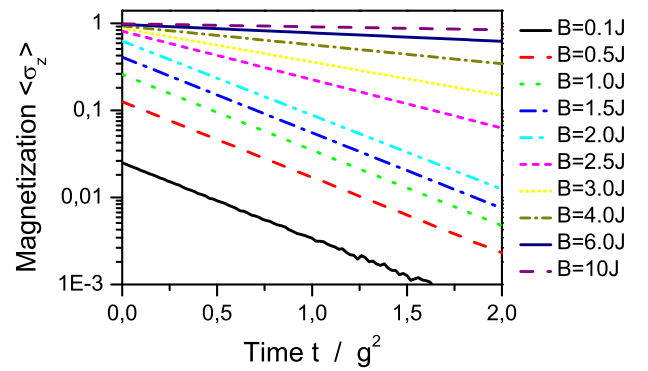


FIG. 5. Evolution of the magnetization $\langle \sigma_z^j \rangle$ in time starting from the system ground state of the XY chain (12), for different magnetic fields B , $g = 0.01 (J/\hbar)^{0.5}$, and $\gamma = 1$. The magnetization decreases exponentially in time.

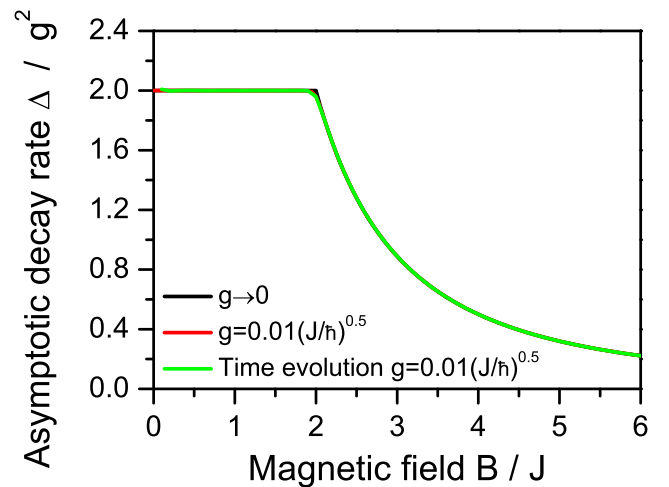


FIG. 6. The ADRs Δ [see Eq. (89)] of the XY chain (12) for $\gamma = 1$ for $g = 0.01 (J/\hbar)^{0.5}$ and $g \rightarrow 0$ (result of perturbation theory) as a function of the magnetic field B are compared with the late-time decoherence rates extracted from Fig. 5. The agreement between the ADR and the late-time decoherence rate shows the validity of our calculations for finite times.

same qualitative behavior is found as in thermodynamic limit, i.e., the ADR signals the quantum phase transition in the system at $B = 2J$. However, finite values of g and N lead to a smearing out of the phase transition.

We have defined the ADR through a diagonalization of the master equation, trying to describe the long-time dynamics of the system. To demonstrate the deep relation between Δ and the dissipative dynamics, we extract the decoherence rate from a dynamical calculation (see Fig. 5). Here we start from the ground state of the system and study the decay of the magnetization in time after the system is brought into contact with a Markovian bath. In this example the exponential decay expected after long evolution times is nicely visible. In Fig. 6 we compare the extracted decay rates for different magnetic fields with the result of the diagonalization. We find an

exact agreement with the asymptotic decay rate numerically calculated with the same finite parameters.

We can calculate the ADR for the XY chain for general values of μ and ν in a similar way. In the spin picture the Lindblad operators are

$$\mathbf{L}^\alpha = g\mu\sigma_\alpha^z = g\mu\frac{i}{2}[c_{\alpha,1}, c_{\alpha,0}], \quad (90)$$

$$\mathbf{L}^\alpha = g\nu\sigma_\alpha^x\sigma_{\alpha+1}^x = g\nu\frac{i}{2}[c_{\alpha,0}, c_{\alpha+1,1}]. \quad (91)$$

We find for the constants in Eq. (79) in the case $\gamma = 1$

$$\epsilon_z/\mu^2 = \begin{cases} \frac{1}{2} & B \leq 2J \\ 1 - \frac{1}{2}\left(\frac{2J}{B}\right)^2 & B \geq 2J, \end{cases} \quad (92)$$

$$\epsilon_x/\mu^2 = \begin{cases} 1 - \frac{1}{2}\left(\frac{B}{2J}\right)^2 & B \leq 2J \\ \frac{1}{2} & B \geq 2J, \end{cases} \quad (93)$$

$$\epsilon/(\mu\nu) = \begin{cases} -\frac{B}{2J} & B \leq 2J \\ -\frac{2J}{B} & B \geq 2J. \end{cases} \quad (94)$$

$$(95)$$

In the symmetric case $\mu = \nu$, the ADR Λ is constant $\Lambda = -4g^2\mu^2$. However, the next larger decoherence rate changes non-analytically:

$$\Lambda_- = \begin{cases} -2g^2\mu^2 \left(3 + \left(\frac{B}{2J}\right)^2\right) & \text{if } B < 2J \\ -2g^2\mu^2 \left(3 + \left(\frac{2J}{B}\right)^2\right) & \text{if } B > 2J. \end{cases} \quad (96)$$

VII. EXPERIMENTAL REALIZATION

We now discuss an experiment suited for the measurement of the asymptotic decoherence rate in spin systems. The quantum simulation of spin systems with trapped ions was proposed in [38], where the spin degree of freedom is represented by two hyperfine levels. The magnetic field can be simulated either by directly driving Rabi oscillations of the hyperfine transition or with position-independent Raman transitions induced by suitably aligned lasers. The spin-spin interaction is mediated via motional degrees of freedoms. State-dependent optical dipole forces (compare with state-dependent optical lattices) are generated by coupling the two hyperfine levels to electronically excited states with off-resonant laser beams. These dipole forces change the distance and consequently the Coulomb repulsion between two ions de-

pendent on their internal states. This state-dependent Coulomb repulsion can be designed to give the required spin-spin interaction. The spin state can be measured by fluorescence imaging of the ions.

In this way the quantum Ising chain [39, 40] and frustrated Ising models [41] have been realized in recent experiments. In these experiments the ions were first cooled to their zero-point motional ground state and optically pumped into a certain spin configuration representing the ground state of the system without spin-spin interactions. Then the spin-spin interactions were adiabatically increased such that the system underwent a phase transition. Finally, it was checked that the final state represented the ground state of the simulated Hamiltonian. A large non-critical 2d Ising system has been simulated with ions in a Penning trap [42]. In the digital approach to quantum simulation with trapped ions, the elements of a general toolbox including Hamiltonian and dissipative dynamics have been demonstrated [43, 44].

We describe in the following how to extend analog quantum simulation to include an incoherent evolution. The Lindblad master equation (3) with Hermitian Lindblad operators $\mathbf{L}^\alpha = g\sigma_z^\alpha$ (see Sec. VB) can be realized by introducing fluctuations of the simulated magnetic field $B^\alpha(t) = B^\alpha + \delta B^\alpha(t)$ [45] as shown in the following. The local magnetic fields $\delta B^\alpha(t)$ should be uncorrelated between different sites $\overline{\delta B^\alpha(t_1)\delta B^\beta(t_2)} = \delta_{\alpha\beta}\overline{\delta B^\alpha(t_1)\delta B^\alpha(t_2)}$. We restrict our derivation to a single Lindblad operator without loss of generality. Let, for example, $\delta B(t)$ constitute a Gaussian stochastic process of zero mean $\overline{\delta B(t)} = 0$ with the time-correlations

$$\overline{\delta B(t_1)\delta B(t_2)} = \begin{cases} \overline{\delta B^2} & \text{for } |t_1 - t_2| \leq T \\ 0 & \text{for } |t_1 - t_2| > T. \end{cases} \quad (97)$$

The correlation time T has to be much shorter than every process in the system (Markovian limit), i.e., $\|\mathcal{H}\|T < \omega T \ll 1$, with the spectral width ω of the Hamiltonian (difference between largest and smallest eigenvalue) and the superoperator \mathcal{H} from Eq. (26). The averaged density matrix evolves like $|\rho(t)\rangle = \overline{\mathcal{U}(t)}|\rho(0)\rangle$, where the bar denotes the statistical average over the fluctuating magnetic field. The time evolution operator $\mathcal{U}(t)$ consists of contributions from \mathcal{H} and

$$\mathcal{V}(t) = i\frac{\delta B(t)}{\hbar}\mathcal{V} = -\frac{i\delta B(t)}{\hbar}(\mathbf{V} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{V}^T). \quad (98)$$

with $\mathbf{V} = \sigma_z$. We can evaluate the statistical average of the time evolution operator in the interaction picture for the superoperators

$$\begin{aligned}
\overline{\mathcal{U}(t)} &= \overline{e^{\mathcal{H}t} \mathcal{T} \exp \left(\int_0^t d\tau e^{-\mathcal{H}\tau} \mathcal{V}(\tau) e^{\mathcal{H}\tau} \right)} \\
&= e^{\mathcal{H}t} \sum_{n=0}^{\infty} \overline{\int_{t \geq t_1 \geq \dots \geq t_n \geq 0} dt_1 \dots dt_n e^{-\mathcal{H}t_1} \mathcal{V}(t_1) e^{\mathcal{H}t_1} \dots e^{-\mathcal{H}t_n} \mathcal{V}(t_n) e^{\mathcal{H}t_n}} \\
&= e^{\mathcal{H}t} \sum_{m=0}^{\infty} \left(-\frac{\overline{\delta B^2}}{\hbar^2} \cdot \frac{T}{2} \right)^m \cdot \int_{t \geq t_1 \geq \dots \geq t_m \geq 0} dt_1 \dots dt_m e^{-\mathcal{H}t_1} \mathcal{V}^2 e^{\mathcal{H}t_1} \dots e^{-\mathcal{H}t_m} \mathcal{V}^2 e^{\mathcal{H}t_m} \\
\overline{\mathcal{U}(t)} &= \exp \left(\mathcal{H}t - \frac{1}{2} \frac{\overline{\delta B^2} T}{\hbar^2} \mathcal{V}^2 t \right)
\end{aligned} \tag{99}$$

with the time ordering operator \mathcal{T} . Between the second and the third line, we keep only even summation indices $m = 2n$ (zero mean Gaussian process), evaluate the statistical average at adjacent times $t_{2n-1} - t_{2n} \leq T$ (correlation time T), and neglect the terms $\exp(\mathcal{H}(t_{2n-1} - t_{2n})) \ll 1$ (Markovian limit). In summary, we have shown that the described fluctuations of the magnetic field generate Markovian dynamics [see Eq. (25)] with Lindblad operators $\mathbf{L}^\alpha = g\sigma_z^\alpha = g\mathbf{V}$ and decoherence strength

$$g^2 = \frac{\overline{\delta B^2} T}{\hbar^2}. \tag{100}$$

In the case of the anisotropic XY chain [see Eq. (12)], the correlation time T is bounded by the width of the single particle excitation spectrum $T^{-1} \gg \max(4B/\hbar, 8J/\hbar)$. In the recent experiment [39] $2J/\hbar \approx B/\hbar = 2\pi \times 4.4$ kHz was used, but experimentally available laser intensities allow $2J/\hbar \approx B/\hbar \approx 2\pi \times 40$ kHz. We propose to create fluctuations of the magnetic field with frequency $T^{-1} = 2\pi \times 1.6$ MHz and variance $\overline{\delta B^2}/\hbar^2 = (0.2B/\hbar)^2 \approx (2\pi \times 8 \text{ kHz})^2$. This would result in the decoherence strength $g^2 \approx 2 \cdot 10^{-3} \text{ J}/\hbar$ and would require coherence times of order $2\pi/g^2 \approx 25$ ms. These coherence times can in principle be achieved in systems of trapped ions [46].

VIII. CONCLUSION

We have investigated the dynamics of open quantum systems with regard to their steady states and asymptotic decay. We have shown that insight into different phases can be gained by spectral analysis of the Liouvillian in analogy to how the spectrum of the Hamiltonian reveals critical behavior in zero-temperature quantum phase transitions.

To illustrate this point we have analyzed in detail the

Liouvillian of open fermionic systems under a translationally invariant, quadratic Hamiltonian, coupled to a Markovian bath. We treat master equations with linear or quadratic and Hermitian Lindblad operators. In both cases, the master equation leads to a closed equation for the covariance matrix from which the steady state covariance matrix and the rates at which it is approached can be obtained exactly (see also [19] for an elegant and comprehensive treatment of both fermionic and bosonic linear open systems and their critical properties and [28] for a detailed study of transport in spin chains under dissipation and dephasing). These results apply as well to a large class of 1d spin systems that can be mapped to quasifree fermions by a Jordan-Wigner transformation. We have proposed an experimental realization of this quantum simulation with trapped ions. Numerical calculations show that our results for the weak decoherence limit do apply to such finite systems.

We focused on the limit of weak decoherence ($g \rightarrow 0$) and showed how to deduce information about critical points from the spectrum of the Liouvillian. In particular, the ADR Δ , i.e., the smallest non-zero eigenvalue of the Liouvillian, can serve as an indicator of phase transitions even if the steady state of the system is trivial and steady state expectation values thus cannot yield such information (as in the case of Hermitian Lindblad operators). Depending on the decoherence process considered, the critical point can be reflected in the spectrum of the system's Liouvillian in the form of a closing gap ($\Delta \rightarrow 0$), a degeneracy of Δ or non-analytic behavior of Δ . These results are summarized in Table I.

With this work we suggest the possibility of detecting certain system properties through an observation of the decoherent dynamics: phase transitions in closed systems can be reflected in non-analytic changes of the ADR [25, 26, 28]. More generally, since the ADR and other decay rates represent physical properties of the system, such non-analyticities can be seen as signature of a transition to a different dynamical regime. This suggests to study

TABLE I. Different dissipative systems studied, characterized by their Lindblad operators and Hamiltonian H . Relevant properties of the ADR Δ and the steady state are listed. x_c denotes critical points of the Hamiltonian H .

Lindblad Op	ADR and Gap	Steady State
Hamiltonian $H = 0$		
$\mu a_\alpha, \nu a_\alpha^\dagger$	gapped, no p.t.	thermal
$\mu a_\alpha + \nu a_{\alpha+1}^\dagger$	gap closes @ $\mu = \nu$, no p.t.	paired
$\frac{i\mu}{2}[c_{\alpha,0}, c_{\alpha,1}],$ $\frac{i\nu}{2}[c_{\alpha+1,0}, c_{\alpha,1}]$	degenerate @ $\mu = \nu$	$\propto \mathbb{1}$
Hamiltonian $H \neq 0$: transl. invariant, critical at x_c		
$\mu a_\alpha, \nu a_\alpha^\dagger$	degenerate @ x_c	$\langle a^\dagger a \rangle$ non-analytic @ x_c
$\frac{i\mu}{2}[c_{\alpha,0}, c_{\alpha,1}],$ $\frac{i\nu}{2}[c_{\alpha+1,0}, c_{\alpha,1}]$	non-analytic @ x_c	$\propto \mathbb{1}$

the phase diagram of steady state correlation functions $\langle A(t)B(t') \rangle$ which will reflect these dynamical transitions.

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